

23rd International Meshing Roundtable (IMR23)

## Towards FEA over Tangled Quads

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### Abstract

In finite element analysis (FEA), a mesh is said to be tangled if it contains an element with negative Jacobian-determinant. Tangling can occur during mesh optimization and mesh morphing. Modern FEA unfortunately cannot handle such tangled meshes, i.e., it will lead to erroneous results. While significant progress has been made on untangling, there are no definitive untangling algorithms.

Danczyk and Suresh recently proposed a theoretical extension to FEA such that one could accurately solve boundary value problems over such tangled meshes. However, their investigation was limited to simplicial meshes.

In this paper we consider the extension of the above framework to tangled quad meshes that pose additional challenges compared to simplicial meshes. These challenges are identified, and a tangling-framework is developed to address explicit quad tangling where quads are allowed to overlap, but are required to be geometrically convex. Numerical examples illustrate the correctness of the proposed framework, opening new opportunities for meshing algorithms.

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*Keywords:* Tangled mesh; inverted elements; FEM

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### 1. Introduction

In modern finite element analysis (FEA), the underlying mesh is required to: (1) be simply connected, (2) conform to the boundary, (3) be of good quality, and (4) not contain inverted elements [14,19]. The focus of this paper is on inverted elements; Figure 1 illustrates a mesh containing an inverted element.

Mathematically, an element is inverted if the determinant of its Jacobian is negative. Furthermore, a mesh containing inverted elements is said to be tangled. Element-inversion and tangling can occur during mesh-generation [12], mesh-optimization [8,9], morphing [17] and large-scale deformation [18].

Modern finite element theory is incapable of handling such tangled meshes. Any implementation based on existing theory will lead to erroneous results. For example, relative error in the order of 1000% can be observed in some cases (see example below). Given such evidence, researchers and practitioners today unanimously recommend untangling

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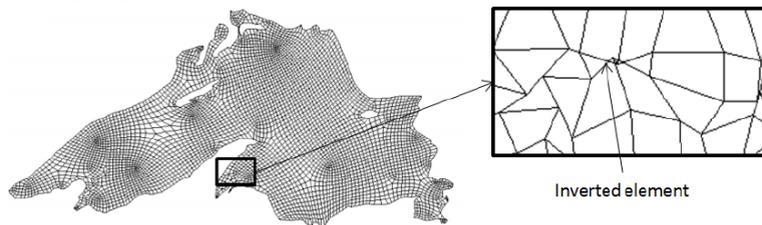


Fig. 1: A quad mesh inverted/tangled elements

prior to analysis. For example, to quote [2]: “Because tangled meshes generate physically invalid solutions, it is imperative that such meshes [be] untangled”.

While there has been significant progress on untangling [10], it can be as difficult as mesh generation [15]. For example, to quote [11]: “Although there is a  $O(n)$  complexity algorithm to detect the presence of tangle elements, there are no known a priori test to determine if a mesh can be untangled”.

To overcome this dilemma, Danczyk and Suresh recently proposed a new theoretical extension to finite element analysis for handling tangled simplicial (triangle and tetrahedral) meshes. The proposed framework offers a novel approach for carrying out accurate finite element analysis over a tangled simplicial mesh [6,7] As an illustrative application of their framework, consider solving a linear elasticity problem over a unit square in Figure 2a. The boundary conditions are given by

$$\begin{aligned} u(x, y) &= 0.1x - 0.2y \\ v(x, y) &= 0.3x + 0.5y \end{aligned} \quad (1)$$

with zero body force. Since the boundary conditions are linear, the exact solution over the entire domain is also given by Equation (1)

If one uses a non-tangled mesh, as in Figure 2a, the exact solution is recovered (to within machine precision) through classic FEA.

On the other hand, consider a tangled mesh in Figure 2b that was artificially created by moving nodes 6 and 7 towards each other. In this configuration, there are six positive elements, two inverted elements, and 15 overlapping pair of elements.

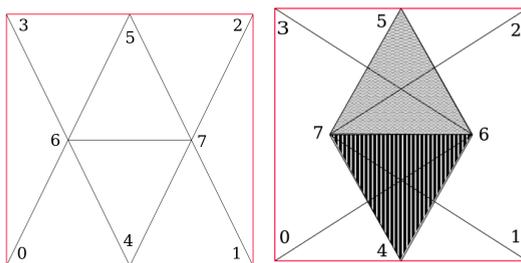


Fig. 2: (a) A non-tangled simplicial mesh, (b) tangled mesh.

The above problem was solved over the tangled mesh using both the classic FEA and tangled FEA, for various configurations of the mesh. As one can observe in Figure 3, using classic FEA, the relative error at node-6 reaches an astounding 3500% in the final configuration. On the other hand, using the tangled FEA proposed by Danczyk and Suresh [6], the error is within machine precision for all positions (barring the singular positions when node 6 coincides with node 7). Additional examples can be found in [6,7].

The objective of this paper is to extend the above tangled FEA framework to *quadrilateral* meshes, since tangling is much more common in such meshes. Our intention is to lay the foundation for FEA over tangled hex meshes, where tangling is a serious problem.

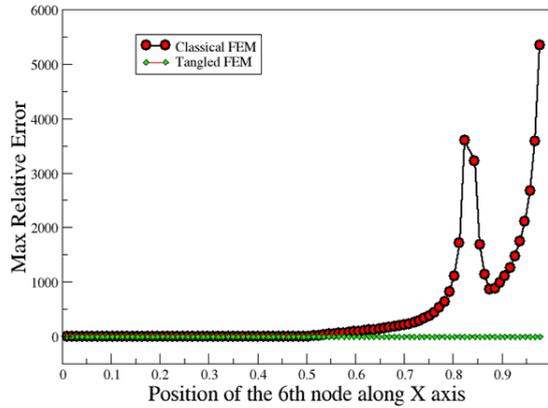


Fig. 3: Relative errors at node-6 using classic-FEA and tangled FEA.

## 2. FEA over Tangled Quad Meshes

We briefly review the basic theory of FEA over quad meshes, followed by an analysis of quad tangling.

### 2.1. Quad Element Shape Functions

Four-noded quad elements, with two degrees of freedom at each node, are used extensively in 2D linear elasticity. The most basic four-node formulation is the bi-linear iso-parametric formulation [5,19] illustrated in Figure 4, where four functions are defined, each associated with one of the four corners:

$$\begin{aligned}
 N_1(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
 N_2(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
 N_3(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
 N_4(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 + \eta)
 \end{aligned} \tag{2}$$

Using these shape functions, one can map a standard point  $(\xi, \eta)$  to a physical point  $(x, y)$  via (see Figure 4):

$$x(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta)x_i \quad y(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta)y_i \tag{3}$$

where  $x_i$  &  $y_i$  are the coordinates of the four nodes.

Given the above transformation, the Jacobian is defined as:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{4}$$

The Jacobian plays a crucial role, for example, in transforming integrals:

$$\int f(x, y) dx dy = \int f(\xi, \eta) |J| d\xi d\eta \tag{5}$$

The inverse of the Jacobian plays a role in transforming gradients:

$$\nabla_{xy}(\cdot) = J^{-T} \nabla_{\xi\eta}(\cdot) \tag{6}$$

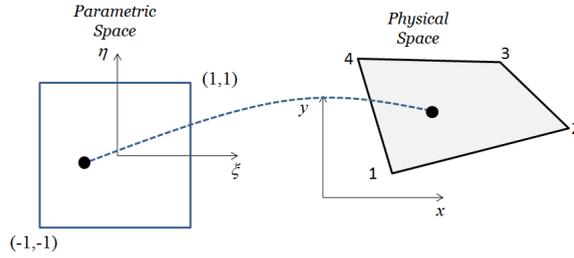


Fig. 4: Isoparametric mapping

Similar to Equation (3), the field over an element is interpolated via:

$$u(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta)u_i^e = \mathbf{N}u^e \tag{7}$$

where  $u_i^e$  are field values associated with the four nodes. Thus, to compute the value of field at a point: (1) first the element the point belongs to is determined, and (2) then the sum of contributions from all four nodes of that element is evaluated.

The field description in Equation (7) satisfies a continuity property in the following sense. Consider, edge 1-2 corresponding to  $\eta = -1$ ; from Equation (2) and (7), we have:

$$u(\xi, -1) = 0.5((1 - \xi)u_1 + (1 + \xi)u_2) \tag{8}$$

Thus the field on this edge depends only the nodal values of 1 and 2. A similar analysis can be established for a neighboring element. It therefore follows that the field is continuous across the entire mesh. Unfortunately, this essential property breaks down when the mesh is tangled

### 2.2. Challenges posed by Tangling

To understand the impact of tangling on the continuity of the field, consider a node and a set of quad elements surrounding it as illustrated in Figure 5a. Let the field value at node-1 be 1, and the value at all other nodes be 0. The resulting field is a hat function illustrated in Figure 5b. From the continuity of the field across element boundaries (see previous Section), it follows that the hat function is continuous. Further, since the global field is simply a linear combination of such hat functions, the global continuity is established.

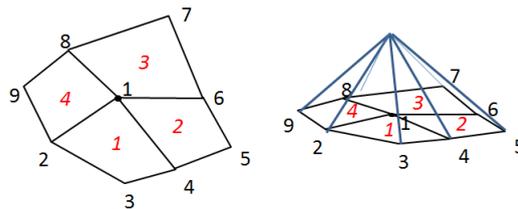


Fig. 5: (a) Non-tangled mesh, and (b) a hat function

Now consider a scenario of tangling where elements 1 & 2 flip over as illustrated in Figure 6b. On the overlapping region, points belong to multiple elements; therefore the hat-function is multiple-valued and is ill-defined. Further, one can easily deduce that the hat function will not be continuous across, say edge 3-4. Thus a fundamental premise of continuous Galerkin FEA is violated, and consequently, erroneous FEA results can be expected.

### 2.3. Self-Tangling of a Concave and Twisted Quads

Figure 6b, illustrated above, is an example of explicit tangling where elements overlap with each other. In addition, quad-meshes can also exhibit *self-tangling* or *implicit tangling*. For example, consider the quad mesh in Figure 7a,

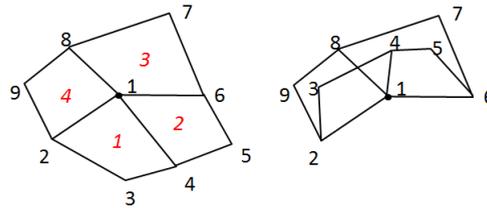


Fig. 6: (a) Non-tangled mesh, and (b) tangled mesh

consisting of five convex quads. By rotating the center quad, the remaining four quads become concave as in Figure 7b. Clearly, *none of the elements geometrically overlap* with each other, i.e., there is no explicit tangling. However, each of the four concave elements is self-tangled for reasons explained below.

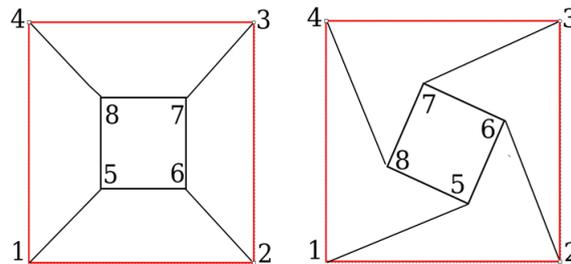


Fig. 7: (a) A valid mesh, and (b) a mesh with concave (self-tangled) elements.

Consider again the bilinear parametric mapping of Equation (3), but onto a concave quad element where the reentrant vertex is located at  $(1/3, 1/3)$  (see Figure 8a). Given the coordinates of this element, one can show via Equation (3) that the parametric mapping is given by:

$$\begin{aligned}
 x(\xi, \eta) &= \frac{5 + 5\xi - 3\eta - 3\xi\eta}{16} \\
 y(\xi, \eta) &= \frac{5 - 5\xi + 5\eta - 3\xi\eta}{16}
 \end{aligned}
 \tag{9}$$

One can now verify that two points  $a(\xi, \eta) = (0, 0)$  and  $b(\xi, \eta) = (2/3, 2/3)$  map to the same physical point  $(5/16, 5/16)$ ; see Figure 8. Thus the parametric mapping is not invertible. This, by itself, poses serious challenges during Gaussian integration. Further, observe that the determinant of the Jacobian for the above mapping is given by:

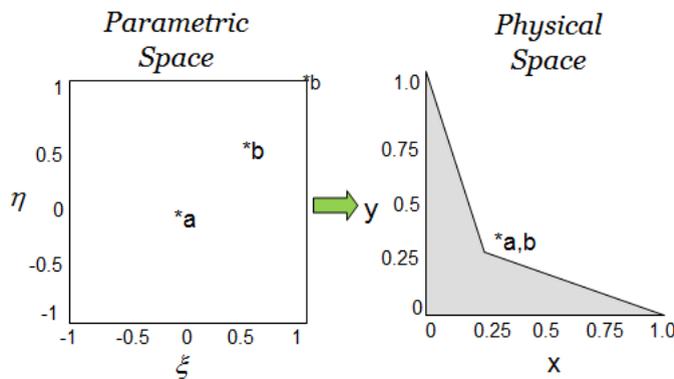


Fig. 8: A simplicial mesh with an inverted element

$$|J| = \frac{2 - 3\xi - 3\eta}{32}
 \tag{10}$$

The determinant vanishes over the line  $2 - 3\xi - 3\eta = 0$ , dividing the parametric space into a + region where the determinant  $|J|$  is positive, and a - region where  $|J|$  is negative (see Figure 9a). The corresponding physical regions are illustrated in Figure 9b and Figure 9c. Observe that these two regions overlap (fold) outside the concave geometry of the quad. *While this fold is visually hidden, it mathematically exists, and it overlaps with the neighboring quad as well!* Similarly, consider the twisted quad illustrated in Figure 10a. One can show that the quad element overlaps with

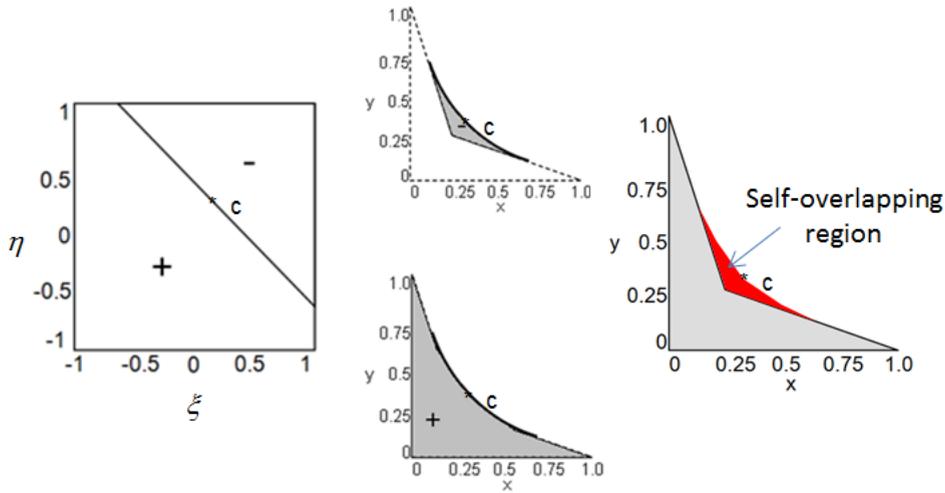


Fig. 9: (a) Parametric space is divided into positive and negative regions, (b) the mapping of the two regions, and (c) final physical space is self-overlapping.

itself (see Figure 10b) and with its neighbors. Self-overlapping is difficult to handle mathematically and numerically.

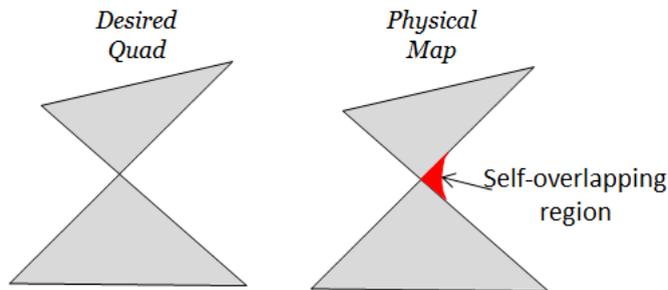


Fig. 10: (a) Intended twisted quad, and (c) self-overlapping region.

It may be possible to handle such elements through non-standard parametric mapping (for example see [4,13]), but this is a topic of future research. Therefore, for the remainder of this paper, we focus on tangled quad meshes where elements may explicitly overlap with each other, but all elements are geometrically convex.

2.4. FEA over Tangled Quad Meshes: Theory

We now address FEA over a tangled quad mesh where all elements are geometrically convex, i.e., the Jacobian determinant  $|J|$  is either entirely positive or entirely negative over each element. The strategy is to arrive at a modified description of the global field that is consistent, unambiguous and continuous over a tangled mesh; this will lead us to the modified FEA formulation for a tangled mesh. Towards this end, we extend the framework proposed by Danczyk and Suresh [6,7], starting with the following definitions.

**Definition 1:** Let  $\theta^k$  be the sign of the Jacobian determinant  $|J|$  for element  $k$ .

**Definition 2:** Let the *index* associated with any point in the mesh be the set of elements the point belongs to

$$I(p) = \{k \mid p \in E_k\} \tag{11}$$

The notion of an index leads naturally to the concept of a cell.

**Definition 3:** A cell is the set of all points with identical index .

Figure 11c illustrates five distinct cells induced by tangling. Observe that a cell need not be convex, or even connected. Further, while there may be other quads that overlap with the four quads shown, we only focus on the quads attached to node-1 since the focus is on a hat function associated with node 1.

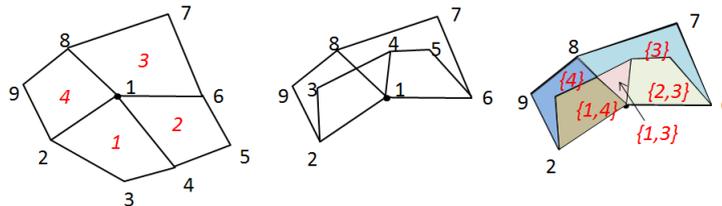


Fig. 11: (a) Non-tangled mesh, (b) tangled mesh, and (c) cells induced.

Now define a cell shape function  $S_\alpha$  over each cell as follows:

$$S_\alpha(p) = \sum_{k \in I_\alpha(p)} \theta^k N^k(p) \tag{12}$$

In the above equation,  $N^k$  is the unique shape function (among the four) that takes a value of 1 at node-1 for element-k. Thus at any point in the mesh, the value of the cell shape function is a linear combination of all the standard element shape functions associated with that cell, weighted by the orientation of the element. Observe that cell shape functions are uniquely defined by construction. Further, the following theorem establishes their continuity.

**Theorem:** The cell shape functions defined via Equation (12) satisfy the following properties: (1) they are continuous across cell-boundaries, and (2) they vanish on the boundary of the cell complex.

**Proof:** The proof is an extension of the one in [6,7] for simplicial mesh. Consider the first part of the proof; let  $C_\alpha$  &  $C_\beta$  be two neighboring cells, with indices  $I_\alpha$  &  $I_\beta$ . It must be proven that the cell shape functions  $S_\alpha(\cdot)$  and  $S_\beta(\cdot)$  are continuous across the common boundary. To this end, consider the difference between the two functions  $S_\alpha - S_\beta$ ; one can group the terms into three categories:

$$S_\alpha - S_\beta = \sum_{k \in I_\alpha \cap I_\beta} [< 1 >] + \sum_{k \in I_\alpha - I_\beta} [< 2 >] + \sum_{k \in I_\beta - I_\alpha} [< 3 >]$$

The first term contains the contribution from every element  $k$  that belongs to both  $I_\alpha$  &  $I_\beta$ . For every such element,  $N_k$  is continuous across the common boundary, therefore the first term vanishes on the boundary, independent of the orientation  $\theta_k$ . Next consider an element  $k$  in the second term, i.e., the element is in  $I_\alpha$  but not in  $I_\beta$ . This implies that in crossing from  $C_\alpha$  to  $C_\beta$ , we must exit element  $E_k$ . Observe that exiting  $E_k$  can only occur in two ways: (1) simultaneously exit  $\omega$ , or (2) enter a neighboring element  $E_j$ . In the first case,  $N_k$  is necessarily zero at the boundary of point of  $\omega$ . Therefore, all such contributions vanish from the second term. In the second case of exiting  $E_k$  and entering one of its neighbors  $E_j$ , there are again two cases: (2a)  $E_k$  and  $E_j$  are of the same orientation, and (2b)  $E_k$  and  $E_j$  are of opposite orientation. If the elements are of the same orientation, then  $E_j$  must belong to  $C_\beta$ , and  $\theta_k N_k = \theta_j N_j$ . Therefore these contributions vanish from the 2<sup>nd</sup> term. Finally, if  $E_k$  and  $E_j$  are of opposite orientation, then  $E_j$  must also belong to  $C_\alpha$ , therefore  $\theta_k N_k + \theta_j N_j = \theta_k N_k - \theta_k N_j$ , which also vanishes since the two element functions are continuous at that point. Through a similar argument, the third term also vanishes. Thus  $S_\alpha(\cdot)$  and  $S_\beta(\cdot)$  are continuous across the common boundary.

Now consider the second part of the proof where it must be shown that if  $C_\alpha$  intersects the boundary of  $\omega$ , then  $S_\alpha$  is necessarily zero on the boundary. Observe that the boundary of any  $C_\alpha$  must be also the boundary of at least one element  $E_k$ . Since we are exiting  $C_\alpha$ , we must also be exiting the element  $E_k$ . Once again, exiting  $E_k$  can only occur

in two ways: (1) simultaneously exit  $\omega$ , or (2) enter a neighboring element  $E_j$ . In the first case,  $iN_k$  is necessarily zero at the boundary of point of  $\omega$ . In the second case  $E_k$  and  $E_j$  must necessarily be of opposite orientation (since we are also exiting  $\omega$ ). As before, the contributions from both elements cancel out at such boundary points, and therefore,  $S_\alpha$  vanishes.

Finally, due to the quasi-disjoint decomposition and continuity, the cell shape functions  $S_\alpha$  can be stitched together to define a continuous hat-function at a node. Since a point belongs to a unique cell, the field is defined via the cell shape function as:

$$u(p) = \sum_{k \in I_\alpha(p)} \theta_k N_k(p) u_k \quad (13)$$

The reader may wish to compare Equations (13) and (7). Thus, to find the value of field at a point in a tangled mesh: (1) find the set of element the point belongs to, and (2) evaluate the sum of contributions from all elements, weighted by the orientation of that element. The global field is now uniquely defined, and is continuous across the entire mesh.

### 2.5. FEA over Tangled Quad Meshes: Linear Algebra

While the concepts of cells and cell shape functions are crucial to establishing the proposed framework, they are not computed explicitly for the following reason. For example, consider the Poisson problem:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} v d\Omega \quad (14)$$

Substituting Equation (13) in above:

$$\int_{\Omega} \nabla \left( \sum_{k \in I_\alpha(p)} \theta^k \mathbf{N}(p) \mathbf{u}_k \right) \cdot \nabla \left( \sum_{j \in I_\alpha(p)} \theta^j \mathbf{N}(p) \mathbf{v}_j \right) d\Omega = \int_{\Omega} \left( \sum_{k \in I_\alpha(p)} \theta^k \mathbf{N}(p) \mathbf{v}_k \right) d\Omega \quad (15)$$

Pushing the gradient inside the summation:

$$\int_{\Omega} \left( \sum_{k \in I_\alpha(p)} \theta^k \nabla \mathbf{N} \mathbf{u}^k \right) \cdot \left( \sum_{j \in I_\alpha(p)} \theta^j \nabla \mathbf{N} \mathbf{v}^j \right) d\Omega = \int_{\Omega} \left( \sum_{k \in I_\alpha(p)} \theta^k \mathbf{N} \mathbf{v}^k \right) d\Omega \quad (16)$$

Since  $\mathbf{N}$  is zero outside the corresponding element  $k$ , we have:

$$\left( \sum_{k \in I_\alpha(p)} \int_{\Omega_k} \theta^k \nabla \mathbf{N} \mathbf{u}^k d\Omega \right) \cdot \left( \sum_{j \in I_\alpha(p)} \int_{\Omega_j} \theta^j \nabla \mathbf{N} \mathbf{v}^j d\Omega \right) = \sum_{k \in I_\alpha(p)} \int_{\Omega_k} \theta^k \mathbf{N} \mathbf{v}^k d\Omega \quad (17)$$

Expanding the terms on the left hand side, and regrouping, we have:

$$\left( \sum_{k \in I_\alpha(p)} \int_{\Omega_k} \theta^k \theta^k \nabla \mathbf{N} \mathbf{u}^k \cdot \nabla \mathbf{N} \mathbf{v}^k d\Omega \right) + \left( \sum_{k, j \in I_\alpha, k \neq j} \int_{\Omega_k \cap \Omega_j} \theta^k \theta^j \nabla \mathbf{N} \mathbf{u}^k \cdot \nabla \mathbf{N} \mathbf{v}^j d\Omega \right) = \left( \sum_{k \in I_\alpha(p)} \int_{\Omega_k} \theta^k \mathbf{N} \mathbf{v}^k d\Omega \right) \quad (18)$$

Since:

$$(\theta^k)^2 = 1 \quad (19)$$

we arrive at the following linear system of equations:

$$(K_{\text{classic}} + K_{\text{overlapping}}) \hat{\mathbf{u}} = f_{\text{oriented}} \quad (20)$$

where

$$\begin{aligned}
 K_{classic} &= \sum_{Assemble} \int_{\Omega_k} \nabla N \cdot \nabla N d\Omega \\
 K_{overlapping} &= \sum_{Assemble} \int_{\Omega_j \cap \Omega_k} \theta^k \theta^j \nabla N^j \cdot \nabla N^k d\Omega \\
 f_{oriented} &= \sum_{Assemble} \int_{\Omega_j} \theta^k N d\Omega
 \end{aligned} \tag{21}$$

The assembly in Equation (21) is the standard finite element assembly, described for example, in [19]

Observe that if there are no overlaps, then Equation (21)b is identically zero. In addition, in a non-tangled mesh, all elements are necessarily positively oriented, and therefore the forcing term in Equation (21)c reduces to the classic forcing term, i.e., classic FEA is exactly recovered. While the above derivation targets 2-D Poisson equation, it can be easily generalized to other elliptic equations. Numerical examples later illustrate the validity the method for 2-D linear elasticity.

## 2.6. FEA over Tangled Quad Meshes: Implementation

Computing the overlapping component of the stiffness matrix in Equation (21) is the non-trivial aspect of tangled FEA. Fortunately, *it only requires a pair-wise intersection of quad elements, i.e., computing cells (as defined earlier) is not required.* Since we assume that quads are geometrically convex, their intersection is always convex (see Figure 12b). The intersection is computed in two stages: (1) use bounding boxes to prune the pairs of quads, and (2) rely on the open-source software CGAL ([www.cgal.org](http://www.cgal.org)) for robust polygon intersection. The intersected polygon is then star-triangulated in order to carry out numerical integration (Figure 12c).

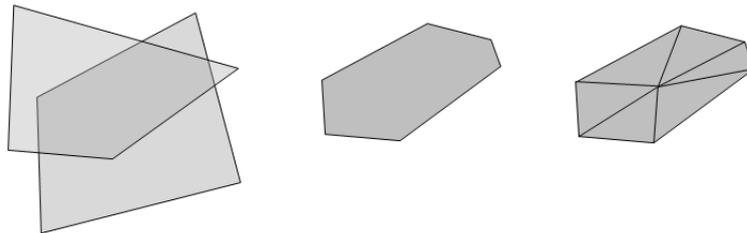


Fig. 12: (a) Overlapping quads, (b) intersection, and (c) star-triangulation.

The overall algorithm to compute the overlapping component of pair-wise element stiffness matrix is described below,

**Algorithm: Compute the overlapping element stiffness matrix**

**Input:** Quads  $j$  and  $k$  that overlap

**Output:**  $K_{overlapping}^{jk}$

1. Find  $\theta_j$  and  $\theta_k$  ( orientation of the two quads)
2. Step 2: Triangulate the intersection polygon
3. Step 3: Set  $K_{overlapping}^{jk} = 0$
4. Step 4: For each triangle  $T$  in the intersecting polygon

- (a) Compute the  $|J|$  ( determinant of the Jacobian) for  $T$
- (b) Choose a set of Gauss points within the triangle
- (c) For each Gauss point  $g$  with weight  $w$

- i. Compute the corresponding point  $(x_j, y_j)$  and  $(x_k, y_k)$  in the parent quads through the inverse mapping.

- ii. Compute the shape derivatives  $\nabla N_j$  and  $\nabla N_k$  at  $(x, y)$
- iii.  $K_{overlapping}^{jk} = w\theta_j\theta_k\nabla N_j \cdot \nabla N_k |J|$

### 3. Numerical Experiments

In all numerical experiments, we solve 2-D linear elasticity problems with zero body force. The material properties are as follows: E (Youngs modulus) = 100 GPa, and (Poissons ratio) = 0.25; the boundary conditions are described under each experiment. The main objective of these experiments is to compare the accuracy of classic FEA against tangled FEA.

#### 3.1. Experiment-1: Triangle Splitting with Analytic Solution

The first experiment mirrors the experiment summarized in Figure 2, except that each triangle is split into four quads (see Figure 13a). The boundary conditions are given by Equation (1) with zero body force. Each side of the square is 1 unit; in Figure 13a nodes 6 and 7 are at 0.25 and 0.75 units respectively. The boundary conditions are such that the exact solution is given by Equation (1). Now consider moving the nodes 6 and 7 to their new positions in Figure 13b (where node 6 is at 0.6 and node 7 is at 0.4), while maintaining the convexity of the underlying triangles and quads. The resulting quad-mesh has all convex elements by construction, but is tangled with six inverted quad elements as illustrated. One can identify and visualize the inverted elements by referring to the numbering of Figure 13a, and tracing the quads in Figure 13b.

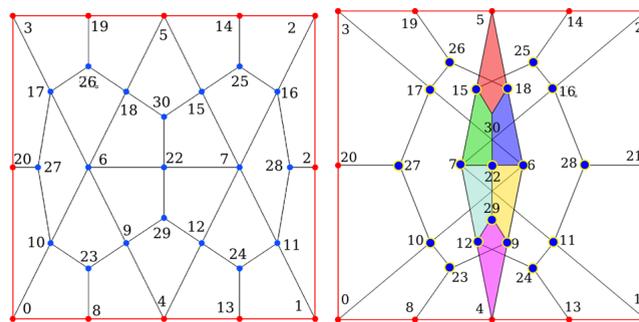


Fig. 13: (a) Non-tangled mesh, and (b) tangled mesh

The above problem (non-tangled and tangled) was solved via classic FEA and tangled FEA for various configurations of node 6 and node 7 as illustrated in Figure 14 (except for the singular position when nodes 6 and 7 coincide). The relative error at node-6 is summarized in Figure 14. Observe that the error in classical FEA reaches an astounding 5000%, while tangled-FEA was found to be accurate to within 0.0001% (machine accuracy could not be achieved; we suspect the source of error to be the inverse mapping).

#### 3.2. Experiment-2: Chord Inversion with Analytic Solution

One can also create tangling by inverting a chain of quads. For example, consider the quad mesh in Figure 15a consisting of four rings of node. One can induce tangling by switching the positions of the 2nd and 3rd ring of nodes, while maintaining the quad connectivity. The resulting tangled quad mesh is illustrated in Figure 15b, where the 2nd layer of quads is now inverted. The inversion can be confirmed by referring to the numbering of Figure 15a, and tracing the quads in Figure 15b.

The above method is now applied to the quad mesh in Figure 16a. Specifically, the 4th layer of quads is inverted via a parameter where corresponds to no tangling, while corresponds to full tangling as in Figure 16b.

For each configuration of the mesh, the exact problem given by Equation (1) was solved via classic FEA and tangled FEA for each configuration of the mesh (except for the singular position where the two chains of nodes coincides).

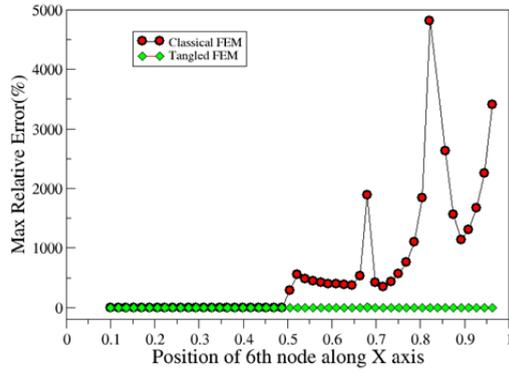


Fig. 14: Numerical results for the problem posed in Figure 13.

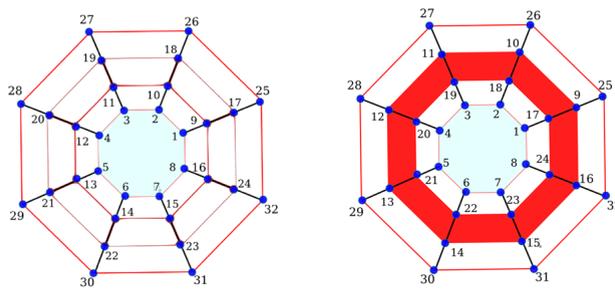


Fig. 15: (a) Non-tangled mesh, and (b) tangled mesh.

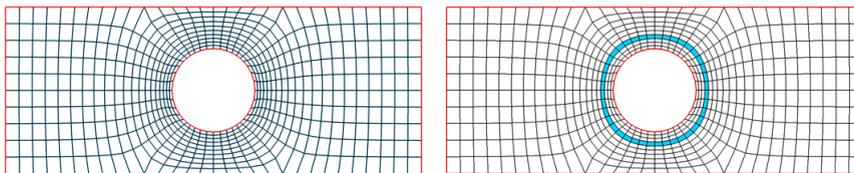


Fig. 16: (a) Non-tangled mesh, and (b) tangled mesh.

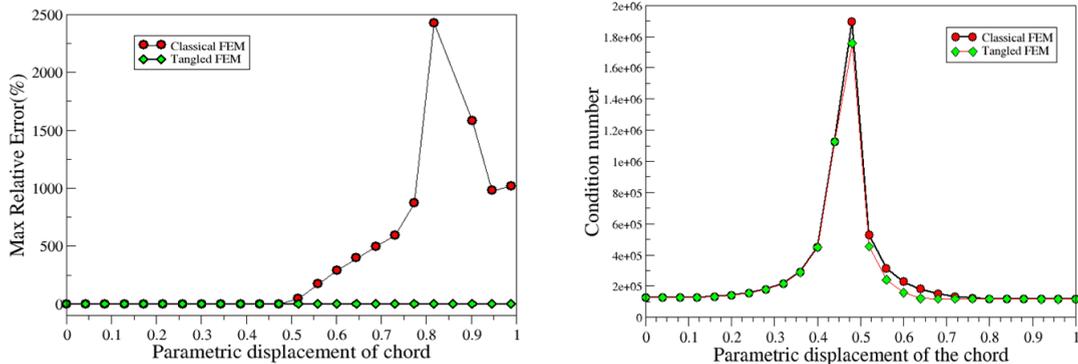


Fig. 17: (a) Numerical results for the problem posed in Figure 16. (b) Condition number for the problem posed in Figure 16.

The relative error is summarized in Figure 17. Once again, observe a highly accurate result for tangled FEA, while classical FEA exhibits significant error with increasing levels of tangling (when).

For the above problem, the condition number of the stiffness matrices 20 was computed for each configuration of the mesh. As one can observe in Figure 17b, the condition numbers for both methods are identical. Further, the condition number blows up as expected since this corresponds to a mesh with degenerate quads.

### 3.3. Experiment-3: Quad Inversion with Non-Analytic Solution

In this experiment, we consider the linear elasticity problem posed in Figure 18a. Unlike the previous problems, this problem has no analytic solution; the expected stress pattern is illustrated in Figure 18b, with a maximum stress of 9.92 Pa.

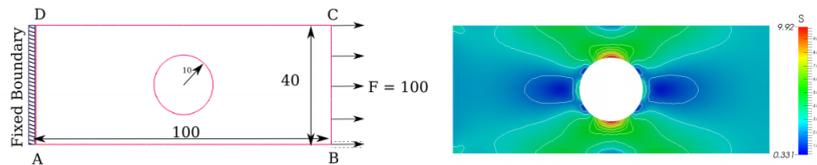


Fig. 18: (a) Tensile loading, and (b) stress distribution.

The starting quad mesh is as in Figure 16a; using the parameter described earlier, the mesh was tangled in a continuous fashion. The problem was solved via classic FEA and tangled FEA for each configuration of the mesh (except for the singular position). The relative error in the maximum stress is summarized in Figure 19. Once again we observe a highly accurate result for tangled FEA, while classical FEA exhibits significant error with increasing levels of tangling.

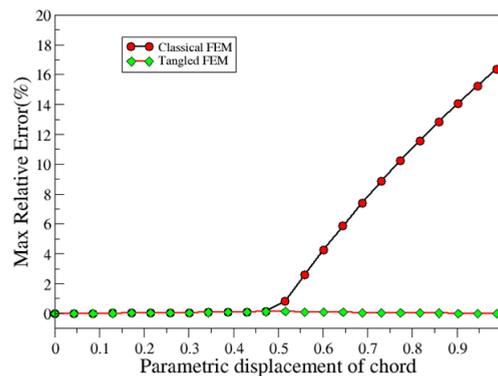


Fig. 19: Numerical results for the problem posed in Figure 18.

## 4. Conclusions and Open Issues

Researchers and practitioners have shunned tangled meshes for decades. In this paper, we have provided a very simple methodology for handling quad tangled meshes within a finite element framework. The proposed methodology can be easily incorporated into classic FEA with minor modifications. Numerical experiments illustrate the correctness of the proposed methodology. While proposed methodology is incomplete (example: extension to concave quad elements and hex meshes), it raises new questions, and opens up new opportunities in mesh generation:

- **Revisiting failed algorithms:** Many promising meshing algorithms such as unconstrained plastering are considered to have failed [16] since either they produce inverted elements or leave a void that cannot be filled with any known regular polyhedra. The proposed framework provides a new opportunity to revisit such algorithms (provided the framework can be extended to hex meshes).

- **Improved mesh quality:** Mesh-quality improving algorithms (example, those implemented in Mesquite [3]) are severely constrained by "element non-inversion". Can mesh-quality be improve significantly if these constraints are removed (this is currently being investigated)?
- **A practical solution to an NP-complete problem:** Deciding whether a 3-dimensional polygon can be triangulated is NP-complete [1] but quoting [1] "If we allow self-intersecting triangulations then computing the optimal triangulation can be done (in cubic time) by applying a simple dynamic-programming procedure". Thus the proposed framework offers a practical means of resolving this NP-complete problem.
- **Reducing software complexity:** The meshing kernel is often a small piece of a meshing-software, but additional codes to check and avoid inversion, intersections etc. significantly add to the software base. Allowing tangling could drastically simplify meshing algorithms.
- **Partial untangling:** While there has been significant progress on untangling, the quality of the untangled mesh may be too poor. One can possibly strike a balance between degree of untangling and mesh quality by relying on the proposed framework to handle a few inverted elements.

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