A dual-representation strategy for the virtual assembly of thin deformable objects

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ABSTRACT

For very large systems of equations arising from 3-D finite element formulation, pre-conditioned iterative solvers are preferred over direct solvers due to their reduced memory requirements. However, in the finite-element analysis of thin-structures such as beam and plate structures, iterative solvers perform poorly due to the presence of poor quality elements. In particular, their efficiency drops significantly with increase in the aspect ratio of such structures.

In this paper, we propose a *dual-representation* based *multi-grid framework* for efficient iterative analysis of thin-structures. The proposed iterative solvers are relatively insensitive to the quality of the elements since they exploit classical beam and plate theories to spectrally complement 3-D finite element analysis. This leads to significant computational gains, as supported by the numerical experiments.

KEY WORDS: Beams, plates, finite element analysis, iterative, pre-conditioners, Euler-Bernoulli, Kirchhoff-Love

1. INTRODUCTION

Thin-structures such as beams, plates and shells find a wide variety of applications across many disciplines including civil, automotive, aerospace, MEMS, etc. Figure 1 illustrates examples of such thin-structures.
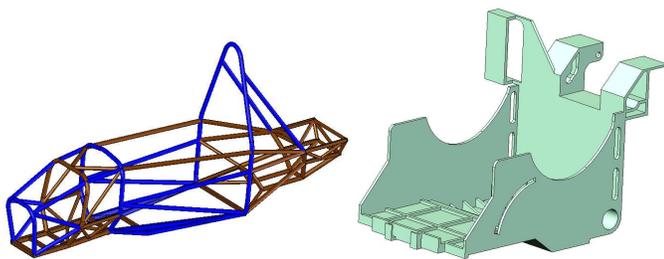
(a) A chassis structure. (b) Printer-housing.
Figure 1: Examples of thin structures.

Currently, three different strategies exist for the structural analysis of such thin structures [1]:

1. <u>Dimensionally reduced analysis</u>: This is the traditional approach where the thin-structure is first dimensionally reduced to 1-D (for beams) or 2-D (for plates and shells). The reduced problem is then solved relying on lower-dimensional theories such as Euler-Bernoulli (for beams) and Reissner-Mindlin (for plates/shells) [2, 3].

Though highly efficient, such methods are hard to integrate within today's 3-D design environment, and with increased geometric complexity, automated construction of reduced models can be cumbersome [4, 5]. Further, the inability of these reduced models to capture stress-concentration and other 3-D effects limits their use in certain applications.

2. <u>3-D finite-element analysis</u>: When accurate stress-prediction is critical and/or when 3-D CAD integration is important, one must rely on a full 3-D structural analysis. Among 3-D methods of analysis, finite element analysis (FEA) is the most popular because of its ability to capture complex geometric features and ease of implementation. However, use of 3-D FEA to analyze thin structures entails a high level of discretization to avoid Poisson-locking and ill-conditioning [1, 6]. This, in turn, leads to a large system of equations [1, 7, 8], that can be computationally challenging to solve. We shall discuss methods for solving such large systems of equations in a later Section as this is the primary focus of this paper.

3. <u>Alternate 3-D methods</u>: Alternate formulations have also been proposed; these include: (1) solid-shell, where relatively low order shape functions

are used across the thickness to overcome ill-conditioning [9, 10]; (2) reduced integration techniques that employ lower-order polynomials for computational efficiency [11-14]; and (3) hybrid or mixed formulations, where both displacement and stresses are used as free variables [1, 12, 15]. Some of the limitations of these methods are discussed, for example, in [16].

In this paper, we propose a fourth strategy that combines the simplicity of dimensionally reduced models and the generality of 3-D FEA:

4. (Proposed) Dual-Representation based 3-D Analysis: In the proposed method, we rely on standard 3-D FEA for assembling the linear systems of equations. However, these equations are solved with the help of classical reduced dimension models. This is achieved by combining the *multi-grid idea* [17, 18] with the recently proposed *dual-representation method* for thin-structure analysis [16, 19-23]. We show that the proposed method leads to rapid convergence, and the computational cost is relatively insensitive to the element quality; this is in contrast to standard pre-conditioners. Further, we illustrate that the proposed method can be combined with standard pre-conditioners to further improve computational efficiency. Finally, since the method is based on 3-D FEA, it can accurately capture 3-D stress-concentration, unlike classical dimension reduction methods.

The remainder of the paper is organized as follows. In Section 2, we present background material related to pre-conditioning and multi-grid. In Section 3, the proposed framework and the underlying concepts of dual-representation are discussed. In Section 4, numerical examples are presented to support the proposed methodology, followed by conclusion and future work in Section 5.

2. BACKGROUND

2.1 Pre-conditioned Iterative Methods

As is well known, accurate 3-D FEA of thin-structures entails high level of discretization, resulting in a large system of equations of the form:

$$Kd = q \qquad (2.1)$$

These large linear systems put severe memory requirements on direct solution techniques [8, 24, 25]. Thus, they are typically solved using Krylov subspace based iterative methods, like conjugate gradient (CG), minimum residual (MINRES), bi-conjugate gradient (BCG), quasi-minimum residual

(QMR), etc [25], together with appropriate *pre-conditioners*.

A pre-conditioner is a matrix or a method that transforms the original system of equations to one that has the same solution, but has more favorable spectral properties [25, 26]. For example, if M is a non-singular matrix that approximates the stiffness matrix K in a spectral sense, then it can be used as a pre-conditioner by multiplying the system in Equation (2.1) with its inverse from the left [24]:

$$M^{-1}Kd = M^{-1}q \qquad (2.2)$$

There are several types of pre-conditioners available today, and amongst them incomplete factorization based pre-conditioners are most widely used; these include incomplete Cholesky (IC) for symmetric positive definite stiffness matrices, and incomplete LU for non-symmetric or indefinite stiffness matrices [24, 25, 27, 28]. Alternate formulations based on *fixed fill-in* or *drop-tolerance* are possible, but require *a priori* prediction of the *fill-in* or *drop-tolerance* parameters, making it difficult to predict the required storage and associated benefits for a given problem [29].

While IC with *no-fill*, i.e., IC(0), can typically reduce the computational cost during iterative analysis, its efficiency drops rapidly as the element quality drops. As a specific example, Figure 2 illustrates the number of iterations required to solve a simple I-beam problem via IC(0) based pre-conditioned MINRES (p-MINRES). For the analysis in Figure 2, number of 3-D elements was kept approximately constant for different aspect ratios, i.e., the element quality dropped with increasing aspect ratio. The iterations were terminated when six orders of magnitude drop in initial residual was achieved.

As one can observe, the computational cost of these *preconditioned* iterative solvers increases rapidly with the aspect ratio of the structure.

Aspect-ratio = Length/Web-thickness

(a) A thin structure problem.
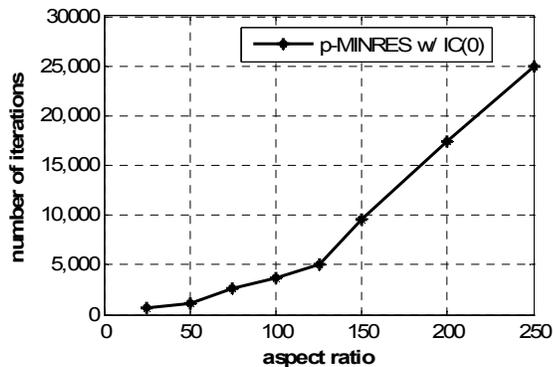
(b) Performance of IC(0).

Figure 2: Performance of IC(0) pre-conditioner.

2.2 The Multi-grid Method

An alternate and powerful 'pre-conditioner' is based on the multi-grid concept. Since multi-grid plays an important role in this paper, we provide a brief overview of the method; additional details can be found, for example, in [18].

In the multi-grid method, the Krylov methods are accelerated by using grids of varying densities. This exploits the fact that low-frequency error-components that are not eliminated by a Krylov 'smoothener' can be projected onto a coarser grid, where it appears as high-frequency, and can easily be smoothed out. The process is repeated until the coarsest grid on which direct solution or smoothing is performed. Thereafter the results are prolonged back up the sequence of grids to the finest level on which the solution is sought [8, 17, 18]. A 2-level multi-grid is depicted in Figure 3.
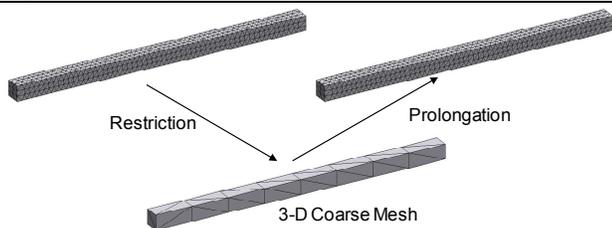
Figure 3: A correction scheme using 3-D coarse mesh.

The stability and efficiency of the multi-grid method depends on the choice of prolongation and restriction operators. Additionally, the performance also depends on the selection of the coarse mesh, and on the algebraic technique used for constructing the coarse stiffness matrix in the case of algebraic multi-grid method [17, 18, 30].

Multi-grid methods have been proposed for linear elasticity problems and for thin-structures [8, 31-35]. However, as mentioned earlier, a coarse mesh based multi-grid is prone to Poisson locking and ill-conditioning, which undermines the quality of the coarse grid correction.

The deterioration in the coarse grid approximation demands more correction steps degrading the performance of the method, especially with decrease in element quality [36]. Hence, an alternate formulation was proposed by Ruge and Brandt [36], which used a composite grid (3-D or 2-D with 1-D) with auxiliary functions specified on 1-D nodes. These auxiliary functions were defined to capture the shear stress at 1-D nodes. The method was limited to straight boundary domains, required use of structured mesh and involved complex computations for 1-D auxiliary functions, making a 3-D implementation fairly difficult; also acknowledged by the authors [36].

A new approach for thin-structure analysis was proposed by Mikulinsky [37], which employed a 1-D mesh for coarse grid correction when solving 2-D or 3-D beam problems using finite difference method. Mikulinsky started with 3-D (or 2-D) equations of elasticity and ignored terms containing derivatives with respect to the thickness-direction coordinate. Though the method was capable of handling curved boundaries, it was limited to boundaries which could be represented by continuous functions, making it difficult to account for features like holes, cuts, stiffeners, etc. Moreover, it required a structured mesh, as in [36], and before applying 1-D correction, the fine mesh had to be coarsened to a level where there were only three nodes along beam thickness, limiting the efficiency of the overall process.

3. PROPOSED METHODOLOGY

3.1 Proposed Concept

The work presented in this paper is based on the idea of using lower dimensional correction. However, instead of explicitly modifying 3-D equations of elasticity as in [37], we use classical dimension reduction theories in conjunction with dual-representation for a one-time construction of lower dimension stiffness matrix, prolongation and restriction operators.

The use of lower dimension physics helps avoid Poisson-locking and ill-conditioning, while dual-representation helps in capturing geometric complexity. The concept is illustrated via a two-level correction step in Figure 4.
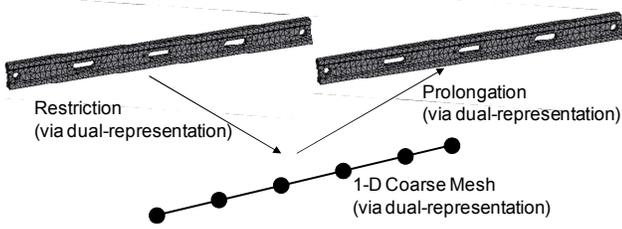
Figure 4: A correction scheme using dual-representation based 1-D beam formulation.

3.2 Dual-Representation: Beams

The dual-representation method for beams as explained in [22], exploits 1-D beam physics, but is implemented within a 3-D environment by appealing to divergence theorem. To understand this, let us consider a 3-D beam of length L and an arbitrarily varying cross-section $A(x)$, oriented along x-direction, as in Figure 2. Also assume that to capture generalized Euler-Bernoulli beam kinematics [3, 19];

$$u(x,y,z) \approx u_0(x) - y\frac{\partial v_0}{\partial x} - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z) \approx v_0(x) \qquad (3.1)$$
$$w(x,y,z) \approx w_0(x)$$

we use a 11 degrees-of-freedom (DOFs) beam element comprising of bending and stretching [19], as in Figure 5, where L_e is the element length.
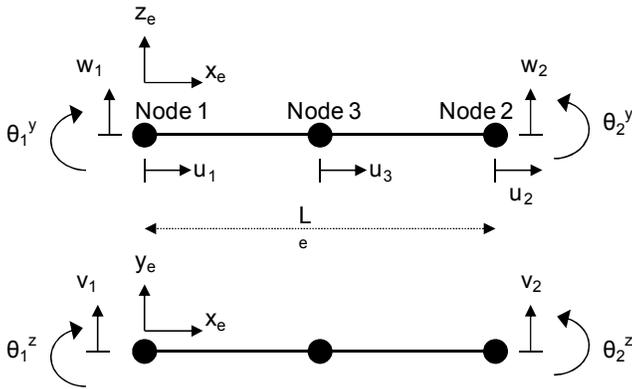
Figure 5: A 11-DOF beam element [21].

If bending is captured via Hermite cubic polynomials [12], $H\left(0 \leq t \leq 1; t = x/L_e\right)$, and stretching via second order Lagrange Polynomials [38], $\ell^2\left(-1 \leq \xi \leq 1\right)$; $\xi = x/L_e$, then the components of Euler Bernoulli beam stiffness matrix can be given by splitting the 3-D domain, say Ω as $[0,L] \otimes A(x)$ [1, 12, 38];

$$\left(K_{EB}\right)_{ij} = \int_0^{L_e} \int_{A(x)} E \left\{ \begin{array}{l} \left(N_{i,x}^u - yN_{i,xx}^v - zN_{i,xx}^w\right)\ldots \\ \ldots\left(N_{j,x}^u - yN_{j,xx}^v - zN_{j,xx}^w\right) \end{array} \right\} dA\, dx \quad (3.2)$$

where $(i, j = 1\ldots13)$ and,

$$N^u = \left[\ell^2(\xi) \quad 0_{1\times8}\right]$$
$$N^v = \left[0_{1\times3} \quad H(t) \quad 0_{1\times4}\right] \qquad (3.3)$$
$$N^w = \left[0_{1\times3} \quad 0_{1\times4} \quad H(t)\right]$$

such that 3-D displacements are approximated as;
$$u_0 \approx N^u d_{1D}; \quad v_0 \approx N^v d_{1D}; \quad w_0 \approx N^w d_{1D} \qquad (3.4)$$

for,

$$d_{1D} = \left\{u_1, u_3, u_2, \quad v_1, \theta_1^z, v_2, \theta_2^z, \quad w_1, \theta_1^y, w_2, \theta_2^y\right\} \quad (3.5)$$

If $A(x)$ is constant or sufficiently simple, then Equation (3.2) can be evaluated analytically. However, with increase in geometric complexity analytical evaluation becomes more challenging.

Instead of splitting the 3-D domain as above, dual-representation method appeals to divergence theorem, which states that for any differentiable vector function $\vec{F}$ [39]:

$$\int_\Omega \nabla \cdot \vec{F}\, d\Omega = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, d\Gamma \qquad (3.6)$$

where $\vec{n}$ is the boundary normal and $\partial\Omega$ is the boundary of Ω. Hence, we need to find a function $\vec{F}$, such that $\nabla \cdot \vec{F}$ is exactly equal to the integrand in Equation (3.2). One such possibility is (see [22]):

$$\vec{F} = E \left[\begin{array}{l} z\left(\begin{array}{l} N_{i,x}^u N_{j,x}^u + y^2\left(N_{i,xx}^v N_{j,xx}^v\right) \\ -y\left(N_{i,x}^u N_{j,xx}^v + N_{i,xx}^v N_{j,x}^u\right) \end{array}\right) \\ +\frac{z^2}{2}\left(\begin{array}{l} y\left(N_{i,xx}^v N_{j,xx}^w + N_{i,xx}^w N_{j,xx}^v\right) \\ -N_{i,x}^u N_{j,xx}^w - N_{i,xx}^w N_{j,x}^u \end{array}\right) \\ +\frac{z^3}{3}\left(N_{i,xx}^w N_{j,xx}^w\right) \end{array} \right] \vec{e}_z \qquad (3.7)$$

where, $\vec{e}_z$ is the unit vector along z-direction.

This gives the dual-representation beam stiffness matrix, as [22];

$$\left(K_{DR-beam}\right)_{ij} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, d\Gamma \qquad (3.8)$$

Thus, for any given geometry, we can obtain a dual-representation beam stiffness matrix using simple boundary integration, where the integration needs to

be performed only on the boundaries where $n_z \neq 0$ (n_z is z-component of the boundary normal) [22]. Also note that for uniform cross-section geometries, dual-representation gives *exactly* the same stiffness matrix as classical reduced dimension formulation, as was shown in [22].

3.3 Dual-Representation: Plates

An analogous dual-representation formulation for 3-D plates can be given, based on Kirchhoff-Love plate kinematics [3, 16];

$$u(x, y, z) \approx -z \frac{\partial \tilde{w}_0}{\partial x}$$
$$v(x, y, z) \approx -z \frac{\partial \tilde{w}_0}{\partial y} \qquad (3.9)$$
$$w(x, y, z) \approx \tilde{w}_0(x, y)$$

If we assume a 2-D discretization based on 12-DOF quad element from [40], such that;

$$\tilde{w}_0(x, y) = S(\xi, \eta) d_{2D} \qquad (3.10)$$

as in [16], we get Kirchhoff-Love plate stiffness matrix as:

$$K_{KL} = \int_\Omega B^T D B d\Omega = \int_\Omega z^2 \tilde{B}^T D \tilde{B} d\Omega \qquad (3.11)$$

where, D is a 3-by-3 constitutive matrix, B is the appropriate strain-displacement matrix [12] and $\tilde{B}$ is defined such that $B = z\tilde{B}$ [16].

Once again exploiting divergence theorem we can re-write the stiffness matrix of Equation (3.11), as;

$$K_{DR-plate} = \int_{\partial\Omega} \left[\frac{z^3}{3} \tilde{B}^T D \tilde{B} \right] \vec{e}_z \cdot \vec{n} \ d\Gamma \qquad (3.12)$$

Please note that for simplicity of expression we have only considered bending in the above plate formulation, however, we can account for stretching in u and v, via Q9 shape functions [12].

3.4 Prolongation and Restriction Operators

Since our correction-step is based on lower dimension physics, the prolongation matrix should be defined in such a way that it aptly maps the lower dimension kinematic constraints to 3-D nodal displacements.

Thus, based on the kinematics of Section 3.2, the displacements for a 3-D beam node (x_i, y_i, z_i) can be given by;

$$u_i = \left(N_i^u - y_i N_{i,x}^v - z_i N_{i,x}^w \right) d_{1D}$$
$$v_i = N_i^v d_{1D} \qquad (3.13)$$
$$w_i = N_i^w d_{1D}$$

This can be used to define a mapping for every 3-D node as;

$$P_{beam}^u(i,:) = N_i^u - y_i N_{i,x}^v - z_i N_{i,x}^w$$
$$P_{beam}^v(i,:) = N_i^v \qquad (3.14)$$
$$P_{beam}^w(i,:) = N_i^w$$

which in turn gives us our beam prolongation and restriction operators as follows [19, 20];

$$P_{beam} = \begin{bmatrix} P_{beam}^u & P_{beam}^v & P_{beam}^w \end{bmatrix}^T; \qquad R_{beam} = P_{beam}^T \qquad (3.15)$$

In a similar manner the prolongation and restriction operators for a 3-D plate node can be constructed. For the kinematics of Equation (3.9), the mapping analogous to Equation (3.14) is;

$$P_{plate}^u(i,:) = -z_i S_{i,x}$$
$$P_{plate}^v(i,:) = -z_i S_{i,y} \qquad (3.16)$$
$$P_{plate}^w(i,:) = S_i$$

which gives the respective plate operators as [16];

$$P_{plate} = \begin{bmatrix} P_{plate}^u & P_{plate}^v & P_{plate}^w \end{bmatrix}^T; \qquad R_{plate} = P_{plate}^T \qquad (3.17)$$

3.5 Algorithm

Typically in a multi-grid setup the number of fine grid iterations pre and post coarse correction are fixed a *priori*. For example, Mikulinsky fixes them at 1 and 2 respectively [37], while various combinations are tested in [31, 35]. However, as suggested in [18], the coarse-grid correction is most effective when high-frequency errors in the fine grid have been eliminated. This forms the basis of our criteria for switching between 3-D and lower dimension mesh.

Suppose we are using a residual minimization Krylov iterative method, like MINRES or QMR. The basis of these algorithms suggests that the residual should drop with every iteration [25]. If the residual stagnates over successive iterations, we can conclude that the iterative method is struggling and needs assistance in the form of coarse-grid or lower-dimension correction, later being the case for our implementation. A similar analogy may be used for energy minimization methods, like CG, using energy stagnation criteria.

For implementation we define a stagnation-parameter (SP) and stagnation-limit (SL). We start with 3-D iterations and apply correction whenever relative residual is less than SP for three consecutive iterations (we call this a 'stagnation point'). If the 3-D residual after correction is less than pre-correction 3-D residual we continue with 3-D iterations, else we reduce SP by some factor, say f, discard correction

and continue with 3-D iterations till new 'stagnation point' is reached. At any point in this adaptive process if SP becomes less than SL, we conclude that the lower dimension correction is not helping 3-D iterative process, hence, stop applying corrections beyond that point. The algorithm is illustrated via a flowchart in Figure 6.
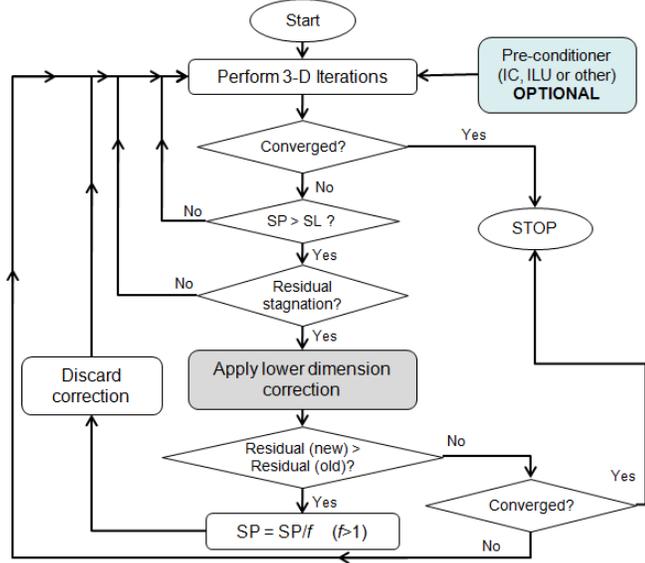
Figure 6: Proposed algorithm.

4. NUMERICAL EXAMPLES

Since the problems considered in this paper lead to symmetric-positive-definite matrices; we shall use incomplete-Cholesky (IC) pre-conditioners for comparison purposes. In particular, we will use IC with *no-fill*, IC(0).

For all numerical examples, we assume that Young's modulus $E = 2 \times 10^{11}$, Poisson ratio $\nu = 0.33$, load per unit area is $q_0 = 1$, $SP = 10^{-3}$ and $SL = 10^{-16}$, unless otherwise stated. Further, we use MINRES iterative method and convergence is assumed when six orders of magnitude drop in residual is achieved.

Unless otherwise specified, a 20 beam element mesh is used for 1-D beam discretization and a 12-by-12 quad element mesh is used for 2-D plate discretization, for construction of corresponding dual-representation 1-D beam or 2-D plate stiffness matrix, as well as prolongation and restriction matrices. *The 3-D model is discretized using sufficiently large number of tetrahedral elements, such that standard FEA does not "lock".*

4.1 Example 1: Element Quality Insensitivity

In this first experiment, we consider a cantilevered rod fixed on the left end, and with a tip load of q_0, on the right, as shown in Figure 7. The 3-D mesh used for this problem has roughly 60,000 DOFs (about 18,000 tet-elements) for all aspect ratios. Thus, with increasing aspect ratio, the element quality drops, with minimum 3-D mesh quality of 0.45 for aspect ratio of 10 and 0.3 for aspect of 50. The convergence plots for various aspect ratios are shown in Figure 7.
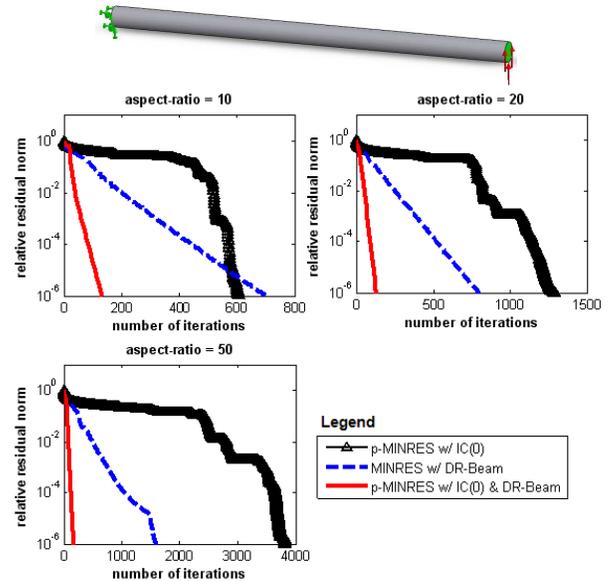
Figure 7: Bending of a rod of different aspect ratios.

It is evident that with increase in aspect ratio the proposed method performs much better than p-MINRES with IC(0), and the combined implementation is nearly insensitive to element quality.

Similar behavior is seen for a uniformly loaded thin-plate with built-in edges, as shown in Figure 8. The 3-D mesh for these examples has approximately 130,000 DOFs (about 37,000 tet-elements, with minimum element quality or 0.3 and 0.21, and average 3-D mesh quality of 0.82 and 0.74).
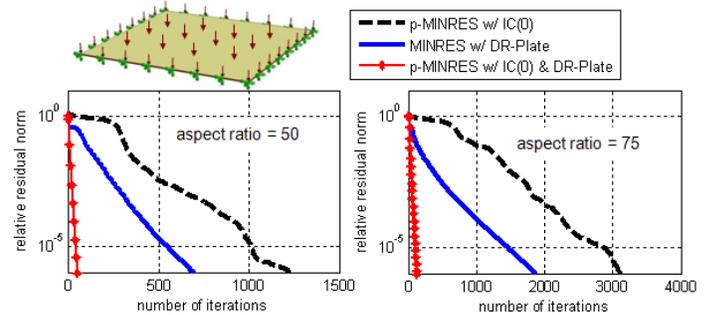
Figure 8: Problem of thin-plate with built in edges.

4.2 Example 2: Complex Geometry

To demonstrate the capability of our method in handling complex geometry, we re-visit the I-beam of Figure 2. Once again 20 1-D beam elements are used for computation of beam stiffness and projection matrices. The number of iterations required for each method is shown in Figure 9. Once again, the advantage of the proposed method is evident. The 3-D mesh used for this case comprises of about 33,500 tet-elements, with average mesh quality dropping from 0.90 to 0.68 with increase in aspect ratio from 25 to 250.
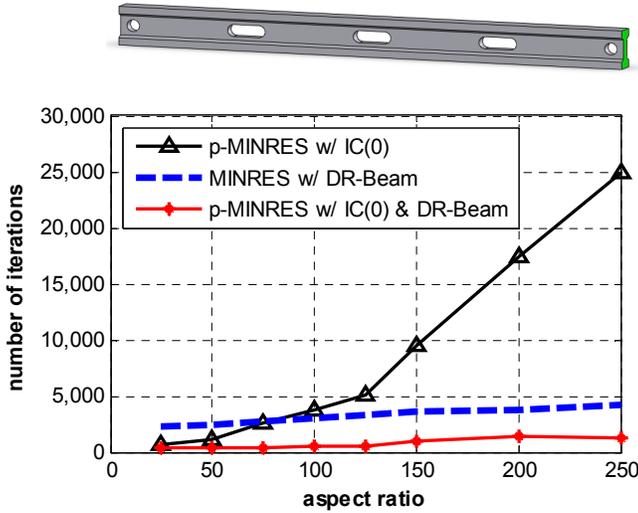
Figure 9: Performance on complex beam geometry.

A plate example having a pattern of rectangular shaped protrusions with filleted sections (on one face) is also presented in Figure 10. It is non-trivial to model such a plate using reduced dimension methods; however, dual-representation captures the impact of features, and leads to rapid convergence as shown in Figure 10. The plate is assumed to have built-in edges and has a uniform load of q_0, on the face without any pattern (illustrated in Figure 10). The 3-D mesh of this problem contained roughly 41,000 tet-elements with 150,000 DOFs and average 3-D mesh quality of 0.76.
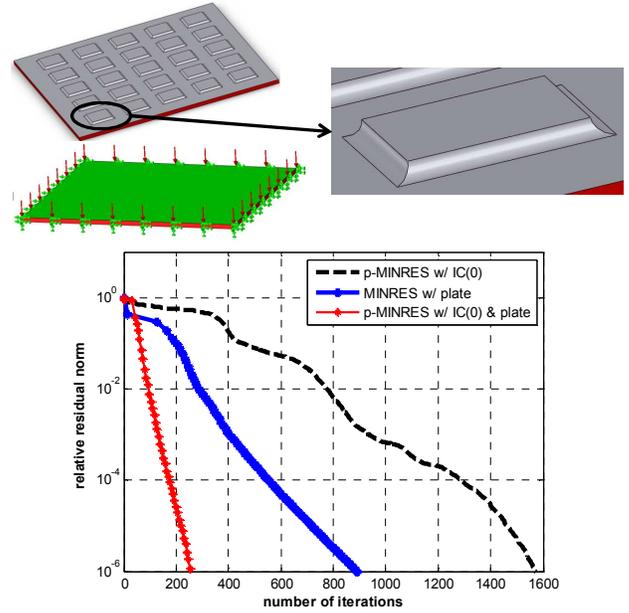
Figure 10: Performance on 3-D plate with features.

4.3 Example 3: Why dual-representation?

To assert the significance of capturing actual 3-D geometry for lower dimension correction, and to highlight the importance of dual-representation in the proposed framework, we consider a *tapered* cantilevered rod, fixed on the left end and having a tip load of q_0 on the right end. For 1-D stiffness matrix formulation, we use two methods; (1) dual-representation (DR), which accounts for actual geometry, and (2) Euler-Bernoulli (EB) assuming average uniform cross-section across the beam length. A comparison of performance of DR and EB based correction is given in Figure 11, for the tapered beam of aspect ratio 10 and 15.
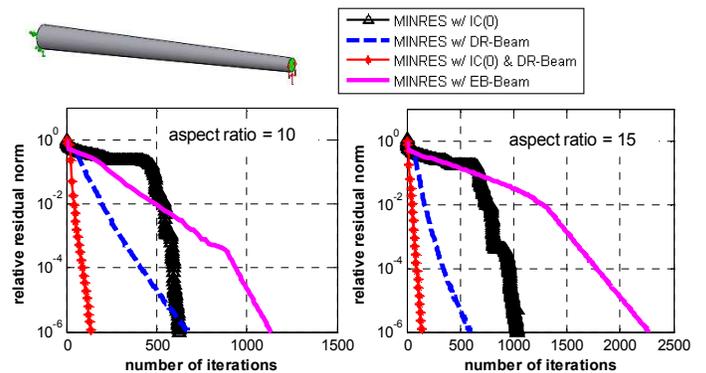
Figure 11: Tapered-rod: comparison of Euler-Bernoulli (EB) with Dual-Representation (DR).

Clearly dual-representation formulation plays a vital role in the correction process. For this example, we used a 3-D mesh with about 32,000 tet-elements,

with an average quality of 0.82 for aspect ratio of 10, and 0.77 for aspect ratio of 15.

4.4 Example 4: Capturing 3-D Stress

Since the proposed method is based on 3-D FEA, it is capable of capturing stress concentration, and it does so more efficiently.

To illustrate this, we consider a tip-loaded t-shaped cantilevered beam, fixed on the left end and having a tip load of q_0 on the right end (similar to that for the problem of Section 4.3). The cross-section of the beam is as shown in Figure 12. The length of the beam was fixed at 200 units, so that the number of iterations required for residual convergence by the methods "p-MINRES w/ IC(0)" and "MINRES w/ DR-Beam" were approximately the same. Approximately 34,000 tet-elements were used to model the problem with minimum element quality of 0.25 and average quality of 0.8.

As illustrated in Figure 12(b), the 3-D stress-error with respect to final 3-D solution drops much faster for the proposed method. For example, after about 600 iterations, the relative error in stress is less than 10^{-4} for the proposed method, while that for the pre-conditioned iterative method is almost 1. Hence, in typical structural analysis applications where stress is the quantity of interest one can terminate iterations much earlier.
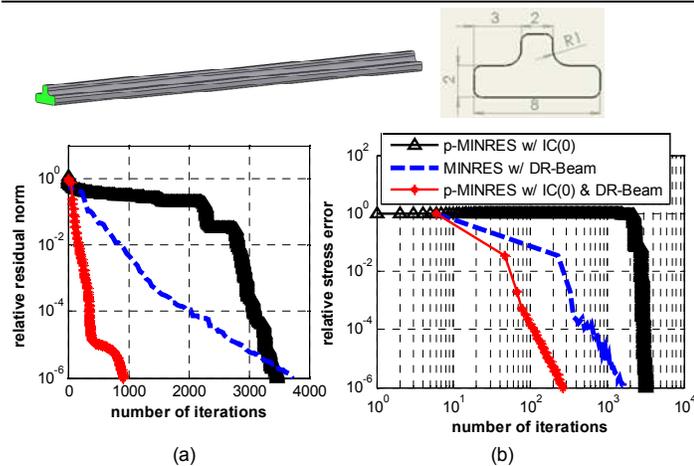
Figure 12: Capturing 3-D von-mises stress error: proposed versus pre-conditioned iterative method (total iterations for convergence ≈ 3,400).

4.5 Example 5: Beam-Structure

The proposed method can further be extended to beam structures, for example, to solve the problem of a 3-D L-shaped solid rod with boundary conditions as in Figure 13(a) (where the ratio of cross-sectional diameter to length for each arm is 20).
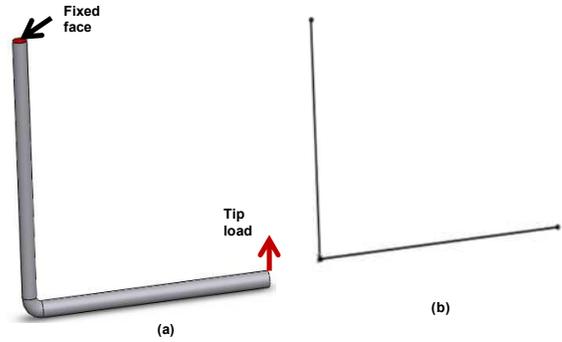
Figure 13: (a) L-shaped rod with curved junction; (b) 1-D model for dual-representation formulation.

For an equivalent 1-D beam model we use a two beam 1-D structure shown in Figure 13(b), and discretize each arm with 4 beam elements. The 3-D discretization had about 40,000 tet-elements with minimum element quality of 0.42 and average quality of 0.83.

The corresponding convergence profile is shown in Figure 14, where we compare p-MINRES with and without dual-representation based beam correction. Use of a curved beam might lead to faster convergence.
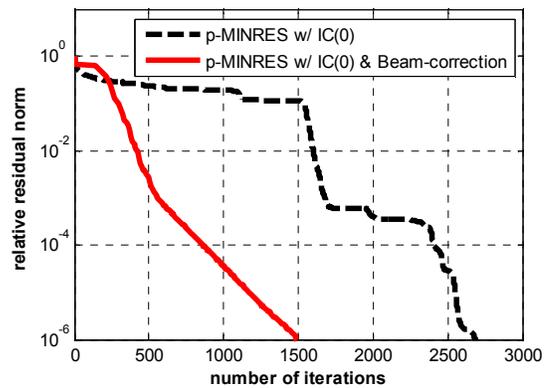
Figure 14: Convergence profile for L-shaped rod.

4.6 Example 6: Computational Efficiency

To illustrate that the proposed method is indeed computationally efficient compared to a standard pre-conditioned iterative solver, in this section, we present in Figure 15 a computational-time comparison for the uniformly loaded plate of Figure 8.

The labels used in Figure 15 are as follows:

- *K-3D*: time taken to assemble the 3-D finite element stiffness matrix computation.
- *IC(0)*: time taken to compute the incomplete Cholesky factorization

- *K-plate*: time taken to compute the dual-representation 2-D plate stiffness matrix, using Kirchhoff-Love plate theory
- *P-plate*: time taken to assemble the projection matrix
- *Iterations*: time taken for the iterative process to converge, using either IC(0), dual-representation or both.

The work was carried out using Matlab R2009b on a 64-bit Windows platform with Intel Core 2 Duo process of 3.17 GHz and 3 GB RAM; when no other process was being executed.
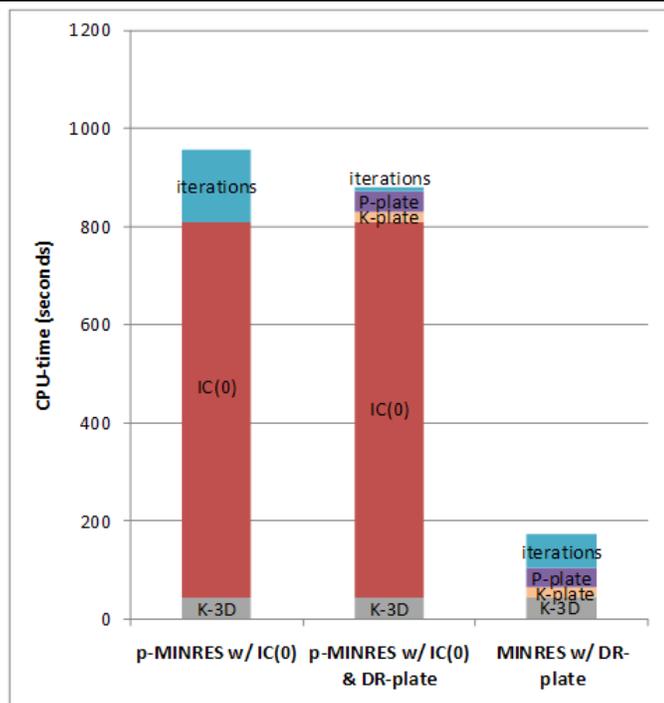
Figure 15: CPU time comparison.

Clearly, the proposed method is much faster than the incomplete factorization scheme, where the bottleneck is the incomplete factorization process. The main reason behind this is that the incomplete factorization is based on some form of Gauss elimination, which involves multiplicative operations on large 3-D matrices; while the dual-representation formulation involves simple linear node-wise summation.

5. CONCLUSION

In this paper, we have described a *dual-representation* based *multi-grid framework* for rapid iterative analysis of thin-structures. The proposed scheme exploits the strengths of a 3-D FEA, as well as lower-dimension models to improve the computational efficiency of standard iterative solvers.

The method as illustrated for beams and plates shows significantly better convergence, especially with drop in element quality. Further, the proposed method can be combined with existing pre-conditioners, resulting in further speed-ups. The proposed method also efficiently captures 3-D stress concentration.

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